RISK ADJUSTMENT FOR LOSS RESERVING BY A COST OF CAPITAL TECHNIQUE

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ABSTRACT. Primarily, Solvency II concerns the amount of capital that EU insurance companies must hold to reduce the risk of insolvency. Determining this risk involves a best estimate of insurance liabilities and an associated risk margin. There are two approaches to determining the risk margin. The percentile approach involves setting a margin above best estimate liabilities so that, to a specified probability, the provisions will eventually prove to be sufficient to cover the run-off of claims. The cost of capital approach determines a risk margin in a way that enables the (re)insurance obligations to be transferred. It involves computing a fair value, which is the amount for which liabilities may be settled, between knowledgeable, willing parties in an arm's length transaction. In this paper we suggest how such a fair value may be computed.

1. INTRODUCTION

Fair value for financial liabilities starts from the principle that the best estimate for the payments from a cash flow is raised with a margin that covers the uncertainty of the ultimate amount of payments. If such an undertaking for payments would be marketable, then it would be easy to define this margin. The Market Value Margin (MVM) is then basically the difference between the price and the best estimate of this future cash flow. But what to do if there is no market available and there is no party to deal with - or to deal with without enormous transaction costs.

Therefore, defining the MVM for Loss Provision of General Insurance is more complex than it seems. This is illustrated by Wüthrich and Salzmann (2010). They calculate the MVM by using a recursive equation. As a consequence of specific assumptions of their loss reserving model, a direct equation for the MVM per accident year is derived. However,

a direct equation for the MVM over aggregated accident years cannot be obtained in closed form in their loss reserving model.

It is not easy to define the MVM even when we look at the prescriptions of the International Financial Reporting Standards (IFRS). IFRS states that a risk margin includes the "maximum amount the insurer would rationally pay to be relieved of the risk that the ultimate cash flows exceed those expected". Actually we find ourselves in the middle of the valuation theory of economics.

One of the core statements for valuation asserts that the value itself will be set by subjective decision criteria between two parties. There is not such a thing as an objective value: an MVM upon loss provisions determined by a general dictated Cost of Capital rate. Objective criteria as prescribed by Solvency II will influence the environment in which these decisions will take place.

In case of an exchange of risk taking it is the insurer, who bears the risk, to deliberate whether or not the price for being relieved of the risk bearing is too high. Next, it is up to the receiving party, i.e. a reinsurer, to accept a price in order to comply with these future obligations. These decisions depend on subjective criteria. The receiving party will take into account its earning capacity: a subjective assessment of the Cost of Capital rate. This gives us the opportunity to calculate at least a minimum transfer value of loss provisions that will comply with the prerequisites of Solvency II.

The key to valuation is found in the amount the receiving party will require as compensation to comply with the ceded undertaking in a satisfactory way - i.e. following the Solvency II regulations. The determination of this value, from the perspective of the receiving party, depends on his subjective earning intensity: the aimed Cost of Capital rate. From this point of view we have integrated Risk Adjusted value of Loss Provision in Integral Financial Modeling (IFM).

2. Risk adjustment by Cost of Capital

In this section we describe our proposal to determine the Risk Adjusted Value (Economic Value) of a given portfolio of insurances, which is the

price a third party may wish to receive in order to take over the risks concerned with the portfolio.

2.1. **General method.** Suppose we have a portfolio of insurances, and we have a method to determine the following two characteristics of this portfolio. First of all, we have estimated a function $b(t)$ which is the *payment intensity* of the claims considered. This means that if we define the nominal payments in the time interval $[t_1, t_2]$ by $B(t_1, t_2)$, then b is given by

$$\mathbb{E}B(t_1, t_2) = \int_{t_1}^{t_2} b(s)\, ds.$$

Note that B is a random process, and we assume that at the present time $t = 0$, we have an estimate of the distribution of B. Secondly, we have estimated a function $V(t)$, which at each time t is defined as the 99.5% quantile of the total future payments. This means that we have the following connection:

$$\mathbb{P}(B(t, \infty) \geq V(t)) = 0.995.$$

The importance of the function $V(t)$ lies in the fact that the insurer would be required by Solvency II regulations to have a minimal reserve equal to $V(t)$ at time t. This part of its capital can therefore not be used to render a profit, other than the risk-free rate.

Other factors that are important to estimate the Risk Adjusted Value of the portfolio are the risk-free growth intensity of capital $\delta_{\mathrm{f}}(t)$ and the (risky) growth intensity $\delta(t)$. The risk-free growth intensity signifies the growth of capital per annum when we invest this capital without any (reasonable) risk of losing the capital. The risky growth intensity is the growth of capital per annum when we invest this capital in order to make money, and this of course entails a risk of losing part of this capital. When we have a capital $E(t)$ and we invest it according to the risky growth intensity $\delta(t)$, we have by definition of the growth intensity that

$$\frac{dE}{dt} = E(t) \cdot \delta(t).$$

The difference between the risky and the risk-free growth intensity is called the Cost of Capital growth intensity $\delta_{\mathrm{C}}(t)$. As described in the introduction, determining the Cost of Capital is mainly a subjective matter. In fact, the price for taking over the insurance risk might be most naturally calculated by using the corresponding Cost of Capital

growth rate of the receiving party.

So suppose a third party, call it a re-insurer, receives X to take over (the risks in) the portfolio. Its starting capital $E(0)$ is then given by X. At time t it will then have a capital $E(t)$. How does this capital change in a small time interval? The re-insurer can invest part of the capital, namely $E(t) - V(t)$, according to the risky rate $\delta(t)$. The reserve $V(t)$ can only be invested according to the risk-free rate $\delta_{\mathrm{f}}(t)$. Furthermore, payments of claims have to be made. Combining this gives us the growth equation for the capital:

$$(2.1) \qquad \frac{dE}{dt} = (E(t) - V(t))\delta(t) + V(t)\delta_{\mathrm{f}}(t) - b(t).$$

We can solve this linear differential equation for E if we know the boundary condition. In order to determine a fair price for the portfolio, it makes sense to take $E(+\infty) = 0$, since then the reinsurer is neutral to the decision whether to take on the risk or not. The solution to (2.1) given this boundary condition is given by

$$(2.2) \qquad E(t) = \int_t^\infty (V(s)\delta_{\mathrm{C}}(s) + b(s)) \exp\left(-\int_t^s \delta(u)\,du\right) ds.$$

It follows that the economic value X, called the Risk Adjusted Value, is given by $X = E(0)$:

$$X = \int_0^\infty (V(s)\delta_{\mathrm{C}}(s) + b(s)) \exp\left(-\int_0^s \delta(u)\,du\right) ds.$$

2.2. **Interpretation.** The formula for the capital $E(t)$ (2.2) has a natural interpretation. Since the re-insurer invests its capital at the risky rate $\delta(t)$, a payment made at time $s \geq t$, so in our setup this would be $b(s)\,ds$, should be discounted using the rate δ. This leads to a discounted total future payment at time t of

$$\int_t^\infty b(s) \exp\left(-\int_t^s \delta(u)\,du\right) ds.$$

Furthermore, reserving $V(s)$ at time s means that the re-insurer cannot invest this capital at the risky rate, but only at the risk-free rate. This corresponds to a *loss* in the time interval ds equal to $V(s)\delta_{\mathrm{C}}(s)\,ds$, since the cost-of-capital rate δ_{C} is exactly the difference between the

risky rate and the risk-free rate. This loss is then discounted at the risky rate, just like the payments, leading to the term

$$\int_t^\infty V(s)\delta_{\mathrm{C}}(s) \exp\left(-\int_t^s \delta(u)\,du\right)\,ds.$$

The sum of these two terms exactly equals the capital at time t.

3. Estimating payments and loss provision

In this section, we provide a brief review of our method to determine a best estimate of loss provision and its associated percentile jointly using paid and incurred triangles. For a more detailed description we refer to Posthuma et al. (2008).

Typically, an insurer will arrange its payments by loss period and development period in a rectangular loss array, also referred to as a run-off table. Since some of the payments lie in the future, this array is not fully observed. The observed part is often referred to as a run-off triangle. We regard the unobserved part of the loss array as a collection of random variables. Then the goal is to determine their probability distributions on the basis of the available data. Naturally, extensive literature exists on this important problem. Perhaps the most widely used approach is the Chain Ladder. Renshaw and Verral (1998) identify the underlying assumptions. Mack (1993) and England and Verral (1999) present ways of estimating the standard error of the prediction. In most cases we have two arrays: an array of payments on settled claims and an array of reservations for claims that have been reported, but not yet settled. Here, we build a joint model for the two arrays. First, we construct marginal models for each array separately. Next, we couple these models by conditioning on the fact that as all claims are eventually settled, the reservations must vanish and the cumulative payments and incurred for a given loss period must become equal.

3.1. **Multivariate normal model.** In this section, we build a joint model for the paid and incurred arrays. First, we construct marginal models for each array separately. Next, we couple these models by conditioning on the fact that as all claims are eventually settled, the reservations must vanish and the cumulative payments and incurred amounts for a given loss period must become equal.

So, we start with two arrays $U_{lk}^{(1)}$ and $U_{lk}^{(2)}$ ($l = 1, 2, \ldots, L$ and $k = 1, 2 \ldots, K$) of independent, normally distributed random variables with means

$$\mathbb{E} U_{lk}^{(1)} = \mu_l \Pi_k^{(1)} \quad \text{and} \quad \mathbb{E} U_{lk}^{(2)} = \mu_l \Pi_k^{(2)}$$

and variances

$$\mathrm{var}\left(U_{lk}^{(1)}\right) = \tilde{\Pi}_k^{(1)} \quad \text{and} \quad \mathrm{var}\left(U_{lk}^{(2)}\right) = \tilde{\Pi}_k^{(2)},$$

where we assume

$$\sum_k \Pi_k^{(1)} = \sum_k \Pi_k^{(2)} = 1.$$

Next, let $Y_{lk}^{(1)}$ and $Y_{lk}^{(2)}$ denote the incremental paid and incurred losses for loss period $l = 1, 2, \ldots, L$ in development period $k = 1, 2 \ldots, K$. Unlike $U_{lk}^{(1)}$ and $U_{lk}^{(2)}$, $Y_{lk}^{(1)}$ and $Y_{lk}^{(2)}$ are *not* independent. In fact, they are coupled in the following way.

Since the total paid over one loss period equals the total incurred over that period, the joint distribution of $Y^{(1)}$ and $Y^{(2)}$ should be such that their row sums are equal with probability one. We can achieve this by specifying the joint distribution of $Y_{lk}^{(1)}$ and $Y_{lk}^{(2)}$ as

$$Y^{(1)} \overset{\mathcal{D}}{=} U^{(1)} | \{U^{(1)}\mathbf{1} = U^{(2)}\mathbf{1}\}$$

and

$$Y^{(2)} \overset{\mathcal{D}}{=} U^{(2)} | \{U^{(1)}\mathbf{1} = U^{(2)}\mathbf{1}\}.$$

where $U^{(1)}\mathbf{1}$ and $U^{(2)}\mathbf{1}$ denote the row sums of the two arrays. This specifies fully the joint distribution of $Y^{(1)}$ and $Y^{(2)}$. Indeed, we can stretch out $Y^{(1)}$ and $Y^{(2)}$ as length KL vectors $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$, respectively and classical theory for multivariate normal distributions tells us that they have a joint multivariate normal distribution. Moreover, we can express the expectation and covariance matrix in terms of the expectation and variances of the $U_{lk}^{(i)}$.

The assumed normal distribution of the entries of the loss arrays is often not appropriate. Occasional large claims result in distributions that are skewed to the right. To account for this skewness the entries are sometimes assumed to have the lognormal distribution. A disadvantage of such a model is the incompatibility of the lognormal distribution with the negative values that do occur in practice in most (incurred) arrays, and the incompatibility of the distribution when aggregating data (the sum of two lognormal random variables is not lognormally distributed).

Also, it will not be feasible to do what we propose – that is, condition on the equality of the row sums of the loss arrays.

We should point out that as a result of the Central Limit Theorem, aggregates of the data will be more normally distributed than the individual entries. We can exploit this fact by aggregating entries in the loss arrays to ensure a closer resemblance to the normal model. We feel that the advantages of the multivariate normal model outweigh those of the multivariate lognormal model.

Stretch out the matrices $U^{(1)}$ and $U^{(2)}$ into the vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$, respectively. As we mentioned before, an advantage of using the normal model is that conditioning poses no problem. Remember that if (Z_1, Z_2) is normally distributed (where Z_1 and Z_2 can be vectors themselves), such that

$$\mathbb{E}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \nu_1 \\ \nu_2 \end{pmatrix} \text{ and Cov}\begin{pmatrix} Z_1 \\ Z_2 \end{pmatrix} = \begin{pmatrix} \Sigma_{11} & \Sigma_{12} \\ \Sigma_{21} & \Sigma_{22} \end{pmatrix},$$

then

$$(3.1) \qquad \mathbb{E}(Z_1|Z_2 = a) = \nu_1 - \Sigma_{12}\Sigma_{22}^{-1}(\nu_2 - a)$$

and

$$(3.2) \qquad \text{Cov}(Z_1|Z_2 = a) = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}.$$

Now let Σ_{11} denote the unconditional covariance matrix of the length $2KL$ vector $\mathbf{u} = (\mathbf{u}^{(1)}, -\mathbf{u}^{(2)})$:

$$(3.3) \qquad \Sigma_{11} = \begin{pmatrix} \text{Cov}(\mathbf{u}^{(1)}) & 0 \\ 0 & \text{Cov}(\mathbf{u}^{(2)}) \end{pmatrix},$$

where $\text{Cov}(\mathbf{u}^{(1)})$ and $\text{Cov}(\mathbf{u}^{(2)})$ are the diagonal covariance matrices of $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$. We use $-\mathbf{u}^{(2)}$ for convenience, since in that case the row sums will be conditioned to add up to zero. Let Σ_{22} denote the covariance matrix of $(U^{(1)} - U^{(2)})\mathbf{1}$. Then

$$\Sigma_{22} = \left(\sum_k \tilde{\Pi}_k^{(1)} + \sum_k \tilde{\Pi}_k^{(2)} \right) I,$$

where I is the $L \times L$ identity matrix. Let $\Sigma_{12} = \Sigma_{21}'$ denote the covariance matrix between $\mathbf{u}$ and $(U^{(1)} - U^{(2)})\mathbf{1}$. Finally, denote $\mathbf{y} = (\mathbf{y}^{(1)}, -\mathbf{y}^{(2)})$.

Since $\mathbb{E}U^{(1)}\mathbf{1} = \mathbb{E}U^{(2)}\mathbf{1}$, Equation (3.1) shows that the (conditional) mean of the vectors $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ is the same as the (unconditional)

mean of the vectors $\mathbf{u}^{(1)}$ and $\mathbf{u}^{(2)}$:

$$\mathbb{E}(\mathbf{y}) = \mathbb{E}(\mathbf{u}|(U^{(1)} - U^{(2)})\mathbf{1} = 0) = \mathbb{E}(\mathbf{u}).$$

However, the vectors $\mathbf{y}^{(1)}$ and $\mathbf{y}^{(2)}$ are of course no longer independent! Equation (3.2) allows us to conclude that the covariance matrix of $\mathbf{y}$, which is equal to the conditional covariance matrix of $\mathbf{u}$ given the event $\{(U^{(1)} - U^{(2)})\mathbf{1} = 0\}$, is given by

$$(3.4) \qquad \qquad \Sigma = \Sigma_{11} - \Sigma_{12}\Sigma_{22}^{-1}\Sigma_{21}$$

This completes the global specification of our model.

Often we do not observe all the elements of the vector $\mathbf{y}$ individually, but compounded in various aggregates. For instance, for certain years we may only have records of payments per quarter, while for other years payments per month are available. Also, we may choose to aggregate payments to make the assumed normal distribution more appropriate. It is also possible that certain entries of the loss arrays are missing for whatever reason.

Suppose we observe J aggregates. If we assume that different aggregates never involve the same payments, we can introduce a zero-one matrix S with pairwise orthogonal rows, of size $J \times 2KL$. Observing various independent sums of the elements of the vector $\mathbf{y}$ then corresponds to

$$\mathbf{z} = S\mathbf{y}.$$

Then $\mathbf{z}$ has a multivariate normal distribution with mean $S\mathbb{E}\mathbf{y}$ and covariance matrix $S\Sigma S'$, where Σ is given in (3.4). The advantage of choosing a multivariate normal model is very prominent here, since in this case it is still feasible to determine the likelihood of the data $\mathbf{z}$.

We obtain estimates of the parameters of our model by penalized maximum likelihood estimation (MLE). Conditionally on the data and the equality of the row sums, the reserve has a multivariate normal distribution and we can use the conditional expectation as a prediction. The uncertainty in this prediction is a combination of the stochastic uncertainty of the model and the uncertainty in the parameter estimates.

4. Implementation issues

The method described above does not directly give estimates for the continuous functions $b(t)$ and $V(t)$. However, we are able to estimate

all payments in future periods, and at the end of each period we can estimate $V(t)$. Define t_i as the last time in period i, taking $t_0 = 0$ as the current date. Define B_i as the expected payments in period i, i.e. $[t_{i-1}, t_i)$. Take $t_n = T$ and suppose that $b(t) = 0$ and $V(t) = 0$ for $t \geq T$. Furthermore, define V_i as the 99.5% quantile of the total future payments to be made after time t_i. We take $V_n = 0$. We can now choose an approximation for the functions b and V:

$$b(t) = \sum_{i=1}^{n} \frac{B_i}{t_i - t_{i-1}} 1_{[t_{i-1}, t_i)}(t) \quad \text{and} \quad V(t) = \sum_{i=1}^{n} V_{i-1} 1_{[t_{i-1}, t_i)}(t).$$

It will usually be natural to approximate the growth intensities by piecewise constant functions as well. Calculating (2.2) will then be a straightforward matter.

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