Stochastic loss reserving for capital management of General Insurance: a Gaussian approach

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1 Introduction

A proper methodology of stochastic loss reserving provides an excellent management tool to identify and manage the value added and the risks of an insurance company. With such a stochastic method calculations for Solvency II or other regulations can be made.

In this paper stochastic loss reserving based on Gaussian assumptions is introduced. Basic properties of the Gaussian distribution facilitate a flexible handling of loss data and projections of future losses. We present a bottom up approach. This approach starts by modeling each line of business in a portfolio separately. In the second step the dependencies between different lines of business are modeled. This allows for the determination of possible diversification between lines of business. Which in turn leads to more accurate capital requirements and risk margins.

Furthermore, a proposal for the specification of mean and covariance functions is made, which allows integral modeling of both reserve risk (past risk), and premium risk (risk in force). This approach makes it possible to optimize the economic value of the portfolio by varying future insurance risk for the underlying lines of business.

The outline of the paper is as follows. First, some properties of Gaussian reserving models are discussed, and a specification of parameters is proposed. Second, more specific models for single lines of business are discussed which can be based on the combined use of paid and incurred data. Third, a method to model multiple lines of business is introduced. This paper will conclude with an example for a portfolio of various lines of business.

This paper presents a unifying framework of foregoing pubications and presentations for actuarial conferences (CAS, ASTIN, GIRO) [1, 3, 4, 2]. These articles and presentations are available on www.posthuma-partners.nl.

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2 Basic building blocks

In this section we explore two basic properties of Gaussian distributions, and remark how their usage is advantageous for loss reserving. Furthermore, the mean and covariance functions are specified. Some new aspects of this specification are the possibility to model future loss periods; the ability to include covariates in the model; and the possibility to model changes in claims handling. Consider a vector $Y \in \mathbb{R}^n$. This vector represents the past and future elements of a runoff table. We assume

$$Y|\mu, \Sigma \sim \mathcal{N}_n(\mu, \Sigma) \tag{2.1}$$

2.1 Properties

Property 1 (Aggregation)**.** *Let A be a linear mapping from $\mathbb{R}^n$ to $\mathbb{R}^m$, then the following relationship holds*

$$AY|\mu, \Sigma \sim \mathcal{N}_m(A\mu, A\Sigma A^\top)$$

Property 2 (Conditioning)**.** *Let Y_1 and Y_2 be constructed of elements from Y. Y_1 and Y_2 are (multivariate) normally distributed with parameters μ_1 and μ_2, and Σ_1 and Σ_2 respectively. Furthermore, let $Cov(Y_2, Y_1) = \Sigma_{21}$. Then the conditional distribution of $Y_2|Y_1$ is given by:*

$$Y_2|Y_1 = a, \mu, \Sigma \sim \mathcal{N}(\mu_2 + \Sigma_{21}\Sigma_1^{-1}(\mu_1 - a), \Sigma_2 - \Sigma_{21}\Sigma_1^{-1}\Sigma_{21}^\top)$$

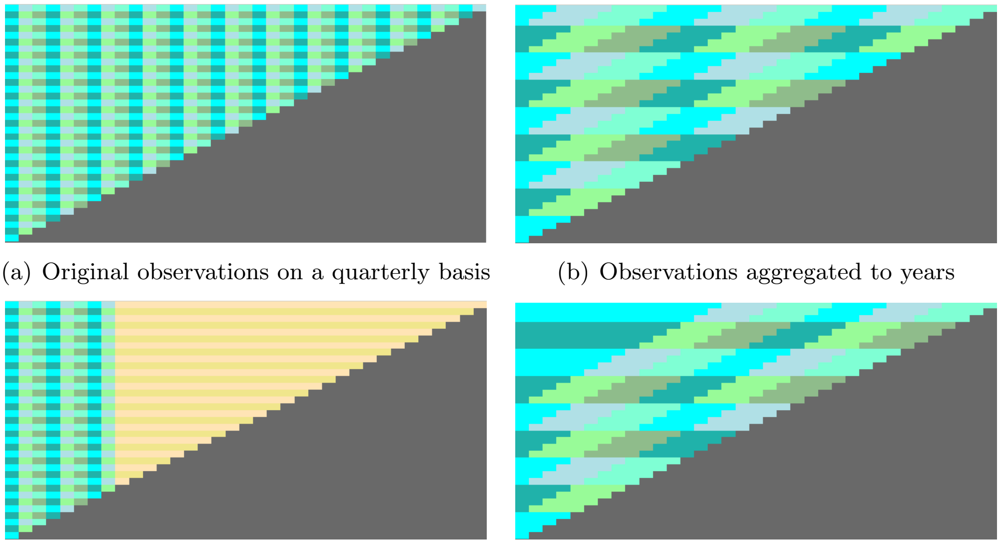
Handling irregular data

Let Y_p be the part of the runoff table which is observed, usually the left-upper triangle, with observed values y_p. Suppose the data is quarterly and is highly irregular over the quarters but stable over the years. Furthermore, there are some missing values, because of a switch from yearly data to quarterly data over the years. Then property 1 can be used to aggregate observations to values per year.

Another issue is when the tail of the triangle, say after two development years, contains a lot of zeros. You do not want these zeros to influence the estimation of the first development periods too much. Property 1 can be used to aggregate the observations on a quarterly basis, and aggregate all observations after two development years together (see figure 1(c)). In figure 1 some examples are shown.

Predictions

Let Y_f be a future value which we wish to predict. Note that because of property 1, this can be any aggregation of future values. For example, the cashflow in a certain year, the total loss reserves, a single cell from the runoff table, etc. Let $\hat{\mu}_p$ and $\hat{\Sigma}_p$ be the estimated parameters of Y_p. With help of property 2 we know the distribution of Y_f.



(a) Original observations on a quarterly basis

(b) Observations aggregated to years

(c) Observations after two development years aggregated

(d) Observations of the first two accounting periods aggregated

Figure 1: *Examples of aggregated observations.*

$$Y_f | Y_p = y_p, \hat{\mu}, \hat{\Sigma} \sim \mathcal{N}(\mu_f, \Sigma_f)$$

Note that the parameters can be estimated on some aggregation of the observations, while the prediction is taken conditional on the original observations.

2.2 Specification of parameters

In the above the parameters μ and Σ are not specified. We further specify a particular cell of a runoff table with l for loss period, and k for development period. The general idea is that the expected value of a particular cell is *ultimate loss* $\times$ *fraction of ultimate loss in development period*. If we let μ_l denote the ultimate loss of loss period l, and $\pi_{l,k}$ the fraction to be paid in development period k, then the expectation of $Y_{l,k}$ is given by:

$$\mathbb{E}(Y_{l,k}) = \mu_l \pi_{l,k} \tag{2.2}$$

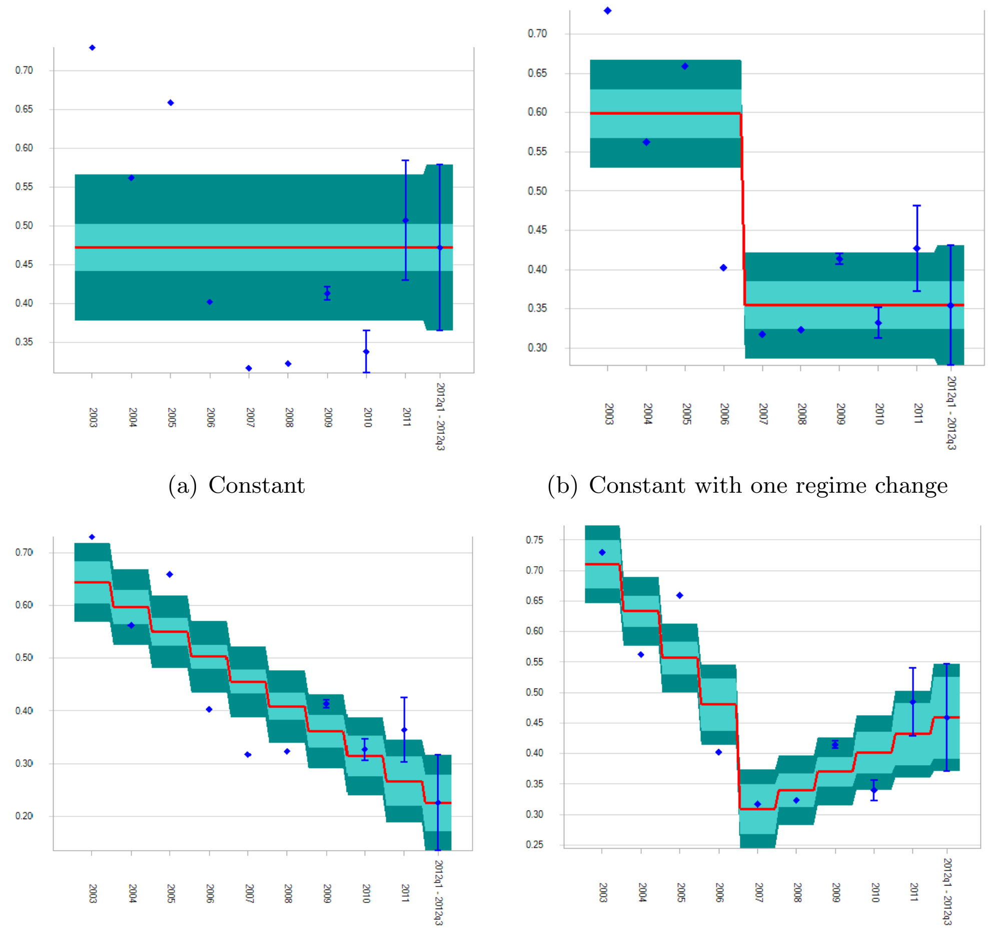
In a similar manner the variance of a single cell is given by:

$$Var(Y_{l,k}) = \nu_l \tilde{\pi}_{l,k} \tag{2.3}$$

Covariances between cells can also be specified, but is not of interest in this section.

Ultimate loss

The ultimate loss of a loss period is loosely calculated as the product *exposure* $\times$ *ultimate loss ratio*. For the ultimate loss ratio a time series is specified over the loss periods via the vector $X\beta$. X gives the structure of the time series, and β are the coefficients. For example X can specify a time series that is constant, linear or has a regime change at some point. See figure 2 for some examples.



(a) Constant

(b) Constant with one regime change

(c) Linear

(d) Linear trend with trend and regime change

Figure 2: *Examples of ultimate loss ratios. The vertical axis shows the ratios, and the horizontal axis the loss periods. The red line shows the estimated time series of the ultimate loss ratio. The medium turquoise region describes estimation uncertainty, and the dark cyan region adds stochastic process uncertainty. The vertical blue lines with centered dots correspond to 66% prediction intervals for the ultimate loss ratio per loss period.*

Another interpretation of X is that it is a matrix of covariates. The vector β then has a regression type interpretation being the relationship of these covariates with the ultimate loss ratios. For example data on macro-economic trends or about the demographics of the policyholders could increase the accuracy of the models. This can also provide useful information for marketing purposes, because it shows which groups have a positive impact on the ultimate loss ratio. These groups can then be targeted in marketing campaigns.

Let w_l denote the exposure of loss period l. To complete we have the following relation

$$\mu_l = w_l f((X\beta)_l)$$

If the exposure of a future loss period is known the model can be extended to generate predictions for this future loss period. Similarly for the variance,

$$\nu_l = \sigma^2 g(w_l) h((X\beta)_l)$$

$f, g, h : \mathbb{R} \to \mathbb{R}$ are functions to be specified for a particular chosen model. σ^2 is a general scaling parameter of the variance.

Development fractions

For a particular loss period l the ultimate loss μ_l has to be paid in the different development periods k. This leads to the following condition $\sum_k \pi_{l,k} = \sum_k \tilde{\pi}_{l,k} = 1 \, \forall l$. Let $b : \mathbb{R}^+ \to \mathbb{R}$ be a function depending on a parametervector θ, such that $\int_0^\infty b(s;\theta)\, ds = 1$. This function models the fraction of the ultimate loss over a small loss period dt which is paid after time s. Then $\pi_{l,k}$ can be modeled by:

$$\pi_{l,k} = \int_0^1 \int_{\max(t,k-1)}^{k} \frac{1}{\alpha_l} b\left(\frac{s-t}{\alpha_l};\theta\right) dsdt \tag{2.4}$$

With this formulation, we keep in mind that if a loss period is one year, and we look in the first year of development, then losses in the second quarter, cannot be paid in the first quarter; for more details, see Section 3.1 of [3]. Also, we allow our function b to be negative, since incremental payments can be negative due to corrections. α_l is a parameter that controls the speed of the settlement. This might be of interest when suspicion exists that due to a change in claims handling, claims are settled faster in later loss periods.

In figure 3 some examplary development curves are shown. This concludes the specification of the parameters. Note that the number of parameters to estimate is reduced drastically in this set-up. In the following section some models based on these building blocks are discussed.

3 Modeling single lines of business

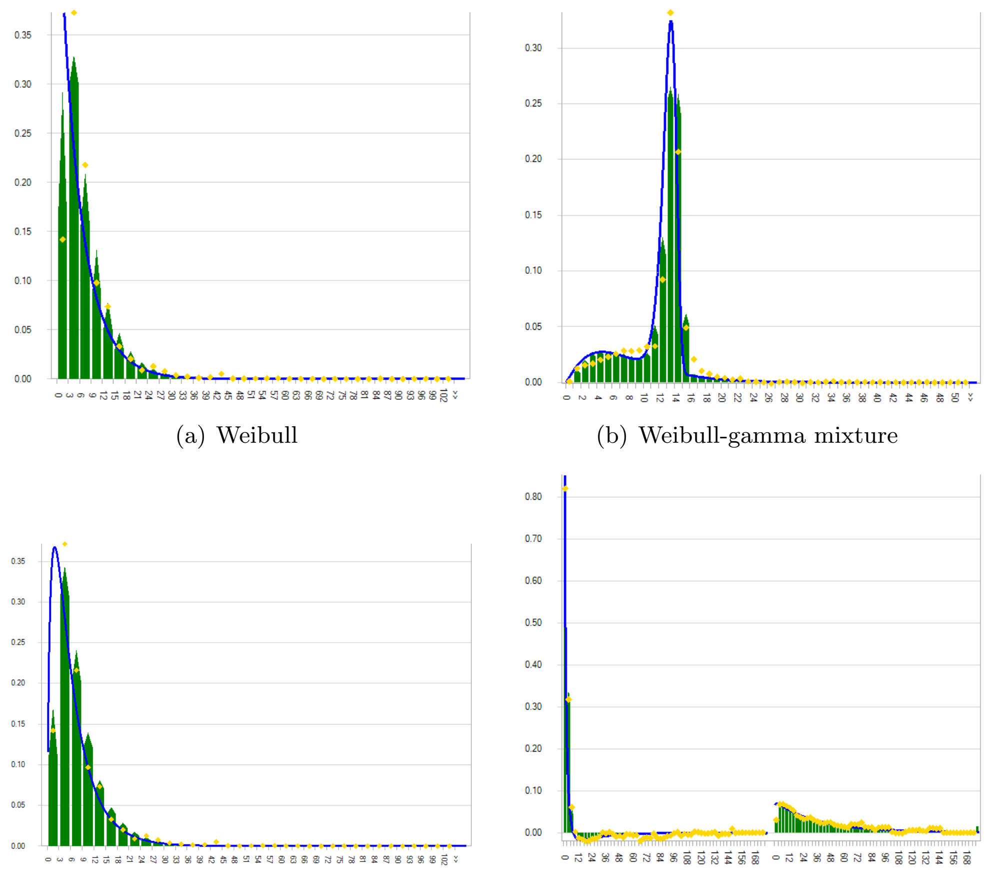
In this section we discuss two models based on the building blocks of the previous section. Both of these models have been widely used in the industry. The first is a model based on single runoff tables, which can be paid, incurred, reported claim numbers, etc. The second is a combined model on paid and incurred.

3.1 Single runoff table

In this model all $Y_{l,k}$ are assumed independent with the following mean and variance:

$$Y_{l,k} \sim N(w_l (X\beta)_l \pi_{l,k}, \sigma^2 w_l^2 \pi_{l,k}) \tag{3.1}$$

Firstly, the standard deviation changes linearly with a change in exposure. This is based on the assumption that when the exposure increases the risks also increase. Secondly, the development



(a) Weibull

(b) Weibull-gamma mixture

(c) Powergamma

(d) Left: incurred development. Right: paid development.

Figure 3: *Examples of incremental development curves. The horizontal axis indicates the development periods, and the vertical axis the fractions. The green bars indicate the discretized development fraction with the triangular shape as an indication of estimation uncertainty. The golden points are the empirical development fractions. The blue lines are the modeled development curves. The powergamma curve 3(c) is a gamma density function with an additional parameter to model an additional point of inflection.*

of expectation and variance is equal. Consequently, the coefficient of variation is decreasing and convex as a function of $\pi_{l,k}$.

3.2 Combined paid - incurred

Let $Z^i_{l,k}$ be an auxilliary variable independently normally distributed with some mean and variance depending on l and k.

$$Z_{i,l,k} \sim N(w_l exp(-(X\beta)_l)\pi_{i,l,k}, \sigma_i^2 w_l exp(-(X\beta)_l)\tilde{\pi}_{i,l,k}) \tag{3.2}$$

i is 1 or 2 for paid or incurred respectively. Here the marginal effect of an increase in the exposure on the standard deviation diminishes.

When the event $R = \left\{\sum_k Z^{(1)}_{lk} = \sum_k Z^{(2)}_{lk} \quad (\forall l)\right\}$ occurs, the sum of changes in case reserves for each loss period equals zero; this corresponds to the fact that eventually, the total incurred loss equals the total paid loss. $X^{(1)}$, and $X^{(2)}$ are assumed to have the following distribution.

$$\begin{aligned} Y_1 &\sim Z_1 \,|\, R \\ Y_2 &\sim Z_2 \,|\, R \end{aligned} \tag{3.3}$$

Following property 2 this is again normally distributed. Hence in this model the information in both the paid and the incurred runoff tables is utilized when making predictions.

4 Modeling multiple lines of business

In section 3 exemplary models based on the building blocks of section 2 for single lines of business are introduced. Most insurance companies have more than one line of business, and will be interested in the diversification effect among the different lines of business. In this section a bottom up method is proposed to establish the degree of diversification among different lines of business. Bottom up here means that it requires as input the estimated parameters of the models of the individual lines of business. Taking that as given, correlation parameters are estimated.

Let $Y_r \in \mathbb{R}^n$ be the cells of the r-th runoff table in a portfolio with $r = 1, \ldots, R$ and $n \in \mathbb{N}$. Assume that the different runoff tables have the same loss and development periods, and those periods are of equal length (we could always enforce this condition by aggregating or splitting up cells where necessary). Each cell contains the loss (paid or incurred) over a certain loss period in a certain accounting period, so we do not consider cumulative run-off tables. Furthermore, let $Y = (Y_1, \ldots, Y_R) \in \mathbb{R}^{Rn}$ be the vector combining the runoff tables of a portfolio. We consider a normal model for the incremental losses:

$$Y \sim \mathcal{N}(\mu, \Sigma) \tag{4.1}$$

The diagonal blocks of Σ are the covariance matrices of the individual runoff tables ($\Sigma_1 \ldots \Sigma_R$). The "off-diagonal blocks", the object of interest here, are the covariance matrices between two runoff tables r and s and are denoted $\Sigma_{r,s}$. For each Σ_r there exists a unique symmetric $n \times n$-matrix B_r such that

$$\Sigma_r = B_r^2 \tag{4.2}$$

B_r can be calculated by eigen decomposition. For any $R \times R$ correlation matrix Γ (i.e., a symmetric positive definite matrix with all 1's on the diagonal), given by

$$\Gamma = \begin{pmatrix} 1 & \rho_{1,2} & \cdots & \rho_{1,R} \\ \rho_{2,1} & 1 & \cdots & \rho_{2,R} \\ \vdots & \vdots & \ddots & \vdots \\ \rho_{R,1} & \rho_{R,2} & \cdots & 1 \end{pmatrix}, \tag{4.3}$$

we now define $\Sigma_{r,s}$ as

$$\Sigma_{r,s} = \rho_{r,s} B_r B_s \tag{4.4}$$

The condition on Γ implies that $\rho_{r,s} \in [-1, 1]$, and $\rho_{r,s} = \rho_{s,r}$.

Claim 1. *Any valid choice of* Γ *will lead to a valid covariance matrix* Σ.

Proof: Since Γ is itself a positive definite matrix, there exist vectors $\gamma_1, \ldots, \gamma_R \in \mathbb{R}^R$ such that

$$\Gamma = \begin{pmatrix} \gamma_1^\top \\ \vdots \\ \gamma_R^\top \end{pmatrix} \begin{pmatrix} \gamma_1 & \cdots & \gamma_R \end{pmatrix} = \begin{pmatrix} \gamma_1^\top \gamma_1 & \gamma_1^\top \gamma_2 & \cdots & \gamma_1^\top \gamma_R \\ \gamma_2^\top \gamma_1 & \gamma_2^\top \gamma_2 & \cdots & \gamma_2^\top \gamma_R \\ \vdots & & \ddots & \vdots \\ \gamma_R^\top \gamma_1 & \cdots & & \gamma_R^\top \gamma_R \end{pmatrix}.$$

Note that since Γ has 1's on the diagonal, $\|\gamma_r\| = 1$ for all $1 \leq r \leq R$. Furthermore, $\rho_{r,s} = \gamma_r^\top \gamma_s$. Now define the $nR \times nR$ matrix

$$B = \begin{pmatrix} \gamma_1^\top \otimes B_1 \\ \vdots \\ \gamma_R^\top \otimes B_R \end{pmatrix}.$$

We then have

$$\begin{aligned} BB^\top &= \begin{pmatrix} \gamma_1^\top \otimes B_1 \\ \vdots \\ \gamma_R^\top \otimes B_R \end{pmatrix} \begin{pmatrix} \gamma_1 \otimes B_1 & \cdots & \gamma_R \otimes B_R \end{pmatrix} \\ &= \begin{pmatrix} (\gamma_1^\top \gamma_1) B_1^2 & (\gamma_1^\top \gamma_2) B_1 B_2 & \cdots & (\gamma_1^\top \gamma_R) B_1 B_R \\ (\gamma_2^\top \gamma_1) B_2 B_1 & (\gamma_2^\top \gamma_2) B_2^2 & \cdots & (\gamma_2^\top \gamma_R) B_2 B_R \\ \vdots & & \ddots & \vdots \\ (\gamma_R^\top \gamma_1) B_R B_1 & \cdots & & (\gamma_R^\top \gamma_R) B_R^2 \end{pmatrix} \\ &= \Sigma. \end{aligned}$$

This proves that Σ is indeed a symmetric positive definite matrix. □

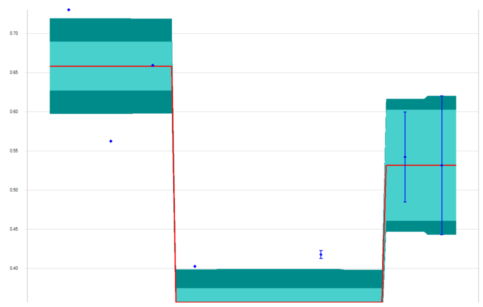
This particular choice of covariance matrix assumes a constant correlation between incremental losses over the same loss period, in the same development period, for the different lines of business. Modeling dependencies is a daunting task, but in this way a reasonable first approximation for the true dependencies is made.

5 Results

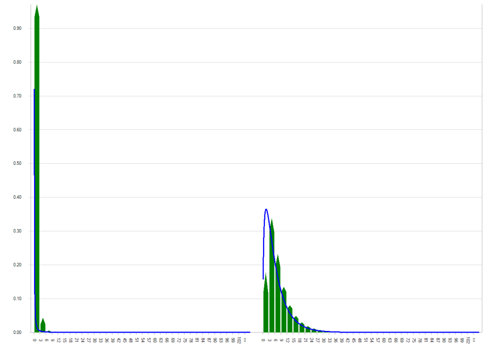
A portfolio of three lines of business (l.o.b.) will be considered: (1) Fire Real Estate, (2) Third Party Liability Motor, and (3) Liability. These three l.o.b. have 2011q4-2012q3 as future loss period for which we have an exposure measure available. Thus we are able to model this future loss period (see section 2.2). In this example the paid-incurred model (see section 3.2) is used to model the three l.o.b. The predictions are calculated using formula (3.2) and the conditioning property 2.

5.1 Estimation results

In figures 4, 5, and 6 the graphs of the ultimate loss ratios and the payment patterns are shown. For detailed explanations of these graphs the reader is referred to figures2 and 3. The l.o.b. TPL Motor and Liability appear to have losses in late development periods. This can be concluded from the uncertainty around the ultimate loss ratios for earlier loss periods, and the relatively large, last fraction (green bar) of losses paid in the development curves.

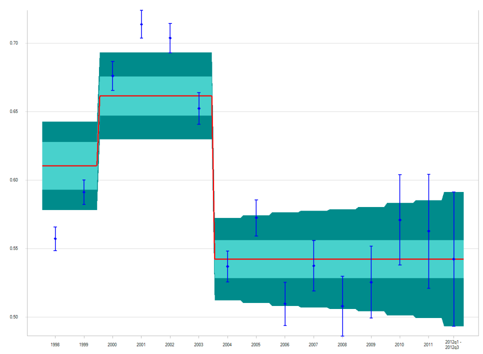


(a) Ultimate loss ratio

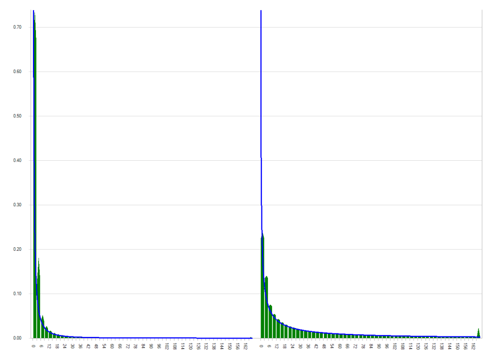


(b) Development pattern of incurred losses (left) and paid losses (right)

Figure 4: *Fire Real Estate*



(a) Ultimate loss ratio



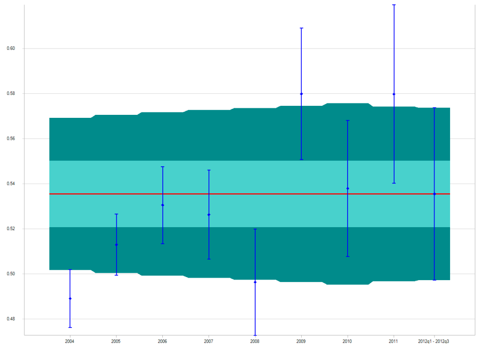
(b) Development pattern of incurred losses (left) and paid losses (right)

Figure 5: *TPL Motor*

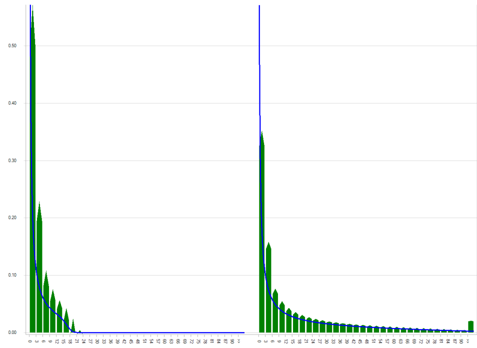
The correlation matrix of this portfolio is estimated to be:

	Fire Real Estate	TPL Motor	Liability
Fire Real Estate	1	-.17	.53
TPL Motor		1	.42
Liability			1

A χ^2-test of the maximum likelihood value against the likelihood value with the identity matrix as correlation matrix gave a p-value of 0.04.



(a) Ultimate loss ratio



(b) Development pattern of incurred losses (left) and paid losses (right)

Figure 6: *Liability*

5.2 Predictions

In table 1 some results of the predictions are shown for the entire portfolio, and for each line of business individually. The table is split up in two parts: reserve risk and premium risk. The reserve risk part shows the results of the past loss periods, i.e. before 2011q3. The premium risk part shows the results of the future loss periods, i.e. 2011q4 -2012q3. The rows of the reserve risk part are:

- Nominal expected value: best estimate of future losses paid.
- Time value of money: discounted future losses paid with the risk-free rate.
- Cost of Capital margin 6%: the Cost of Capital margin calculated as demanded by Solvency II regulations. This margin requires the projection of 99.5th percentiles and expected values of future losses paid until runoff. For more information on the Cost of Capital margin the reader is referred to [2].
- Risk Adjusted Loss (RAL): the loss reserve taking into account the time value of money and the Cost of Capital margin.
- 90th percentile: 90th percentile of discounted future losses paid.

The rows of the premium risk part are:

- RAL: the risk adjusted loss of both past and future loss periods calculated integrally. That is, taking into account diversification between loss periods.
- Risk premium for future losses: measure of the risk present in the losses of the future loss paid. The calculation method used here might shift the risk too much towards the past loss periods.
- Premium income: the premium to be received for the future loss period. This is used as the exposure measure for (2011q4 -2012q3).
- Risk % of premium: what percentage of the premium is to be allocated as a liability on the balance sheet when market value based accounting is used.

Note that these results could not be obtained when using simpler loss reserving methods like chain ladder. That is, the prediction of payments of future loss periods; the calculation and prediction of realistic 90th or 99.5th percentiles; and the calculation of the Cost of Capital margin could not be done with chain ladder.

		(in 1,000 euro)	**Total**	**Fire**	**TPL Motor**	**Liability**
Reserve risk	Nominal expected value	(1)	30,178	2,661	21,832	5,685
	Time value of money	(2)	(2,095)	(23)	(1,787)	(285)
	Cost of Capital margin 6%	(3)	5,716	134	4,542	1,040
	RAL	(4) = (1)+(2)+(3)	33,799	2,772	24,587	6,440
	90th percentile	(5) on (1) + (2)	30,660	2,883	21,888	5,889
Premium risk	RAL	(6)	50,830	8,294	30,449	12,087
	Risk premium for future losses	(7) =(6)-(4)	17,031	5,522	5,861	5,647
	Premium income	(8)	30,000	10,000	10,000	10,000
	Risk % of premium	(9) = (7)/(8)	57%	55%	59 %	56%

Table 1: Portfolio results

In figure 7 it is shown what happens to the 90th percentile of future losses paid and the RAL (both on reserve and premium risk) when the premium income is not the estimated 10,000 euro per line of business, but 9,000 or 11,000. It can be seen that these changes hardly have an impact on the RAL, and some impact on the 90th percentile.

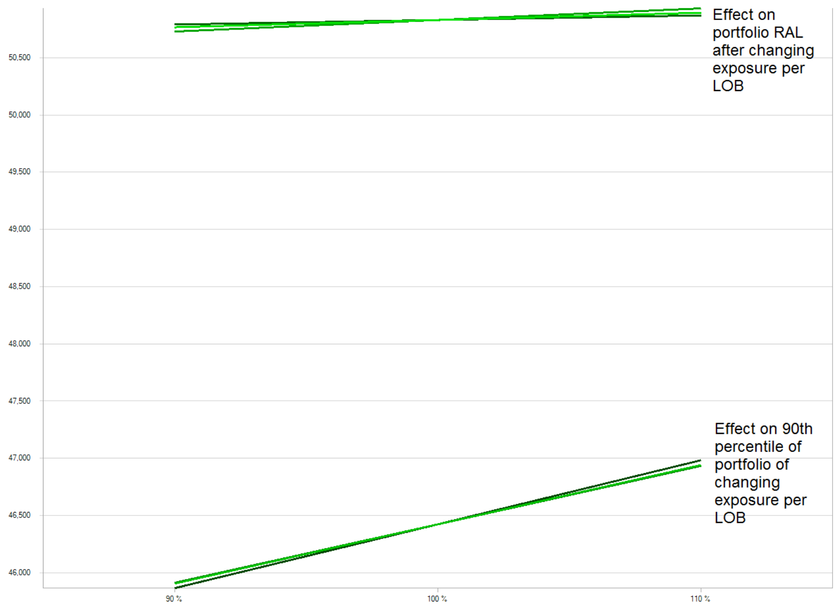


Figure 7: *Exposure uncertainty. What happens if the future exposure is x% of 10,000 euro per line of business?*

Note that, premium income = number of policyholders * premium per policyholder. It is assumed that a change in the number of policyholders has no impact on the premium per policyholder, and that the additional policyholders is a homogeneous risk to the existing policyholders. This could for instance be achieved by increasing marketing efforts through the same or similar channels. Under these assumptions, the company should try to attract more policyholders, because all l.o.b. are profitable (see table 1). This will increase the premium income, but would have little impact on the RAL (see figure 7).

6 Conclusion

In this paper a stochastic loss reserving model based on the multivariate normal distribution was introduced. The use of this distribution has several advantages: (1) the multivariate normal distribution is closed under linear transformations; (2) the predictive distribution is also a normal distribution.

The mean and covariance were chosen to be functions of exposure measures, ultimate loss ratios, and development fractions. This has the advantage that future loss periods can be included, and that actuarial judgment can be translated in the models.

The model given was extended to a paid-incurred model, which uses both the incurred observations and the paid observations. Afterwards, the model was extended to a full portfolio model which gives a bottom-up approach to model an entire portfolio. In the last section, an example of a portfolio of three l.o.b. was given.

References

[1] E. A. Cator and V. Lous. Modeling multiple runoff tables. *presented at ASTIN The Hague 2013*, 2013.

[2] B. Posthuma, E. A. Cator, V. Lous, and E. van Zwet. Risk adjustment for loss reserving by a cost of capital technique. *Presented at GIRO 2011, Liverpool*, 2011.

[3] B. Posthuma, E. A. Cator, W. Veerkamp, and E. van Zwet. Combined analysis of paid and incurred losses. *Casualty Actuarial Society E-forum Fall 2008*, 2008.

[4] P. ter Berg. Nonlinear normal correlated loss array. *ASTIN Topic: Dynamic Financial Analysis*, 2001.